Problem Set 1 Solutions – Relativity (GOI36A)

1 Basics of Index Notation

1.) a.) Allowed. The indices μ and ν each only appear once, and so they are free indices. They both can take any value from 0 to 3, so this is a $4 \times 4 = 16$ component object.

b.) Not allowed. μ appears twice upstairs, which is not allowed according to Einstein summation notation.

c.) Not allowed. ν appears twice downstairs, which is not permitted in Einstein summation notation.

d.) Allowed. ν appears twice, once upstairs and once downstairs, so we sum over it with no problems. μ and ρ each only appear once, so they are free indices. This is a $4 \times 4 = 16$ component object.

2.) To check if an equation is allowed, we have to make sure that the free indices (i.e. the ones *not* summed over) match on both sides of the equation. Importantly, both the label and the position of the index must match!

a.) Allowed. ν appears once upstairs and once downstairs, so it's summed over, while the index μ is free on both the left-hand and right-hand side. Also, it appears upstairs on both sides of the equation, and so the index structure matches. There is one free index μ, so this is a 4 component equation.

b.) Not allowed. μ is a free index on both sides of the equation, but it appears downstairs on the left-hand side and upstairs on the right-hand side, so the two sides of the equation do not match in index structure.

c.) Allowed. μ is summed over, while ν is free. ν appears upstairs on both sides of the equation. There is only one free index, so this is a 4 component equation.

d.) Not allowed. μ is the free index on the left-hand side, while ρ is the free index on the right-hand side. They are different indices, and thus independent of one another, so this equation does not make sense.

e.) Not allowed. While it's true that the free index μ matches on both sides, the index ν appears three times on the left-hand side, which violates our index notation rules.

3.) The equation is valid. There are no free indices on either side of the equation, and all other indices appear twice (once up, once down) and are thus summed over. There are no inconsistencies with Einstein summation notation, and so such an equation is allowed.

When we sum over indices, we often call them "dummy indices", because their labels don't mean anything, as they are summed over anyways. This means we are free to relabel all dummy indices. So, on the right-hand side, we can relabel ρ by μ and σ by ν. The equation then becomes

$$T_{\mu\nu} A^\mu B^\nu = T_{\mu\nu} A^\mu B^\nu \ . \tag{1.1}$$

This is a trivial equation; it's like saying $x = x$, which is obviously true. So, this equation does not tell us anything useful.

2 Index Gymnastics

4.) Using the metric tensor to lower indices, we find

$$\begin{aligned}
T_\mu{}^\nu &= \eta_{\mu\rho} T^{\rho\nu} \ , \\
T^\mu{}_\nu &= \eta_{\nu\rho} T^{\mu\rho} \ , \\
T_{\mu\nu} &= \eta_{\mu\rho}\eta_{\nu\sigma} T^{\rho\sigma} \ .
\end{aligned} \tag{2.1}$$

Now, remember that $\eta_{00} = -1$, $\eta_{11} = \eta_{22} = \eta_{33} = 1$, and all other components are zero. So, we find that

$$\begin{aligned}
T_0{}^0 &= \eta_{0\rho} T^{\rho 0} = \eta_{00} T^{00} + \cancel{\eta_{01}} T^{10} + \cancel{\eta_{02}} T^{20} + \cancel{\eta_{03}} T^{30} \\
&= \eta_{00} T^{00} = -T^{00} \ .
\end{aligned} \tag{2.2}$$

Similarly, the other ones are given by

$$T_{10} = -T^{10} \ , \quad T_0{}^1 = -T^{01} \ , \quad T_1{}^1 = T^{11} \ , \quad T^1{}_1 = T^{11} \ . \tag{2.3}$$

5.) To solve this, we first simply note that $A_\mu = \eta_{\mu\nu} A^\nu$, or equivalently that $A^\mu = \eta^{\mu\nu} A_\nu$. (Remember, $\eta^{\mu\nu}$ is the inverse of $\eta_{\mu\nu}$, but both matrices look identical in form.) Similar relations hold for the vector B^μ. This means that we can write

$$A^\mu B_\mu = (\eta^{\mu\nu} A_\nu) B_\mu = A_\nu (\eta^{\mu\nu} B_\mu) = A_\nu B^\nu \ . \tag{2.4}$$

Since ν is a dummy index we are free to relabel it to μ, and we are left with $A^\mu B_\mu = A_\mu B^\mu$.

6.) **Bonus:** This solution is essentially exactly the same as the previous problem. We just have to first note that we can use the metric tensor to lower indices, which means that we can write

$$S^{\nu_1 \ldots \nu_{i-1}}{}_\rho{}^{\nu_{i+1} \ldots \nu_n} = \eta_{\rho\sigma} S^{\nu_1 \ldots \nu_{i-1} \sigma \nu_{i+1} \ldots \nu_n} \ . \tag{2.5}$$

Similarly, we can also lower the index of the other tensor in exactly the same way:

$$T^{\mu_1 \ldots \mu_{i-1}}{}_\rho{}^{\mu_{i+1} \ldots \mu_n} = \eta_{\rho\sigma} T^{\mu_1 \ldots \mu_{i-1} \sigma \mu_{i+1} \ldots \mu_n} \ . \tag{2.6}$$

With these two expressions in mind, we can now manipulate the left-hand side of the given equation and show that the position of the contracted index can be flipped:

$$\begin{aligned}
T^{\mu_1 \ldots \mu_{i-1} \rho \mu_{i+1} \ldots \mu_n} S^{\nu_1 \ldots \nu_{i-1}}{}_\rho{}^{\nu_{i+1} \ldots \nu_n} &= T^{\mu_1 \ldots \mu_{i-1} \rho \mu_{i+1} \ldots \mu_n} \left(\eta_{\rho\sigma} S^{\mu_1 \ldots \mu_{i-1} \sigma \mu_{i+1} \ldots \mu_n} \right) \\
&= \left(\eta_{\rho\sigma} T^{\mu_1 \ldots \mu_{i-1} \rho \mu_{i+1} \ldots \mu_n} \right) S^{\mu_1 \ldots \mu_{i-1} \sigma \mu_{i+1} \ldots \mu_n} \\
&= T^{\mu_1 \ldots \mu_{i-1}}{}_\sigma{}^{\mu_{i+1} \ldots \mu_n} S^{\mu_1 \ldots \mu_{i-1} \sigma \mu_{i+1} \ldots \mu_n} \ .
\end{aligned} \tag{2.7}$$

Dummy indices can be freely relabeled, and so if we relabel σ to ρ in the last line, we are left with exactly the desired result. Also, note that it didn't matter what rank tensors T and S we had, nor which pair of indices between them was contracted. Thus, you can see that in general it doesn't matter which index is up and which is down when any two indices are contracted between arbitrary tensors.

7.) The sum of the antisymmetric and symmetric pieces is simply given by

$$T^{(\mu\nu)} + T^{[\mu\nu]} = \frac{1}{2}\left(T^{\mu\nu} + T^{\nu\mu} + T^{\mu\nu} - T^{\nu\mu} \right) = \frac{1}{2}\left(T^{\mu\nu} + T^{\mu\nu} \right) = T^{\mu\nu} \ . \tag{2.8}$$

8.) To prove this, we first note that

$$T^{(\nu\mu)} = \frac{1}{2}\left(T^{\nu\mu} + T^{\mu\nu}\right) = \frac{1}{2}\left(T^{\mu\nu} + T^{\nu\mu}\right) = T^{(\mu\nu)} \ , \tag{2.9}$$

and so we are free to swap indices around in this symmetric piece. Similarly, the antisymmetric piece satisfies

$$T^{[\nu\mu]} = \frac{1}{2}\left(T^{\nu\mu} - T^{\mu\nu}\right) = -\frac{1}{2}\left(T^{\mu\nu} - T^{\nu\mu}\right) = -T^{[\mu\nu]} \ , \tag{2.10}$$

and so we pick up a minus sign when we swap indices in the antisymmetric piece. By swapping the indices in both $T^{[\mu\nu]}$ and $S_{(\mu\nu)}$, we will pick up an overall minus sign:

$$T^{[\mu\nu]}S_{(\mu\nu)} = (-T^{[\nu\mu]})(S_{(\nu\mu)}) = -T^{[\nu\mu]}S_{(\nu\mu)} \ . \tag{2.11}$$

But, since both μ and ν are dummy indices, we are free to relabel them. In particular, we will send $\nu \to \mu$ and $\mu \to \nu$, and we are left with

$$T^{[\mu\nu]}S_{(\mu\nu)} = -T^{[\mu\nu]}S_{(\mu\nu)} \ . \tag{2.12}$$

The only solution to this is $T^{[\mu\nu]}S_{(\mu\nu)} = 0$.

9.) Using the result from problem 6, we can rewrite the expression we are interested in as

$$T_{\mu\nu}A^{\mu}A^{\nu} = T_{[\mu\nu]}A^{\mu}A^{\nu} + T_{(\mu\nu)}A^{\mu}A^{\nu} \ . \tag{2.13}$$

Since multiplication is commutative, we are free to swap the order in which we multiply A^{μ} and A^{ν}, and so $A^{\mu}A^{\nu} = A^{\nu}A^{\mu}$. This means that $A^{\mu}A^{\nu}$ is *symmetric* in its indices μ and ν. And, as we showed in problem 7, any symmetric tensor contracted with an antisymmetric tensor vanishes. Therefore,

$$T_{[\mu\nu]}A^{\mu}A^{\nu} = 0 \ . \tag{2.14}$$

Equation (2.13) then simply reduces to

$$T_{\mu\nu}A^{\mu}A^{\nu} = T_{(\mu\nu)}A^{\mu}A^{\nu} \ . \tag{2.15}$$

10.) The trace of $\hat{T}^{\mu\nu}$ is:

$$\hat{T} = \eta_{\mu\nu}\hat{T}^{\mu\nu} = \eta_{\mu\nu}T^{\mu\nu} - \frac{1}{4}\eta_{\mu\nu}\eta^{\mu\nu}T = \left(1 - \frac{1}{4}\eta_{\mu\nu}\eta^{\mu\nu}\right)T \ . \tag{2.16}$$

Recall that $\eta^{\mu\nu}$ is defined to be the inverse of $\eta_{\mu\nu}$, such that their contraction is the identity matrix:

$$\eta_{\mu\rho}\eta^{\rho\nu} = \delta^{\mu}_{\nu} = \begin{cases} 0 & \mu \neq \nu \\ 1 & \mu = \nu \end{cases} \tag{2.17}$$

The symbol δ^{μ}_{ν} is called the Kronecker delta symbol (which you should think of as the identity matrix). Using the fact that the metric tensor is symmetric, we conclude from this relation that

$$\eta_{\mu\nu}\eta^{\mu\nu} = \eta_{\mu\nu}\eta^{\nu\mu} = \delta^{\mu}_{\mu} \ . \tag{2.18}$$

We must sum over all values of $\mu = 0, 1, 2, 3$ in this expression. At each of these values, the Kronecker delta symbol will be 1 (since its indices are equal), and so we find that

$$\eta_{\mu\nu}\eta^{\mu\nu} = \delta^{\mu}_{\mu} = \delta^{0}_{0} + \delta^{1}_{1} + \delta^{2}_{2} + \delta^{3}_{3} = 1 + 1 + 1 + 1 = 4 \ . \tag{2.19}$$

Plugging this back into (2.16), we find that

$$\hat{T} = \left(1 - \frac{1}{4} \times 4\right)T = 0 \ , \tag{2.20}$$

and thus the trace of $\hat{T}^{\mu\nu}$ vanishes.

11.) **Bonus:** In d spacetime dimensions, everything we discussed above will be the same except for the trace of the Kronecker delta symbol. The index μ will take values $\mu = 0, 1, \ldots, d-1$, and so $\delta^\mu_\mu = d \times 1 = d$ instead of just 4. So, if we define $\hat{T}^{\mu\nu}$ by

$$\boxed{\hat{T}^{\mu\nu} \equiv T^{\mu\nu} - \frac{1}{d}\eta^{\mu\nu}T} \, , \tag{2.21}$$

then the trace of this tensor is given by

$$\hat{T} = \eta_{\mu\nu}\hat{T}^{\mu\nu} = \left(1 - \frac{1}{d}\eta_{\mu\nu}\eta^{\mu\nu}\right)T = \left(1 - \frac{1}{d}\delta^\mu_\mu\right)T = \left(1 - \frac{d}{d}\right)T = 0 \, . \tag{2.22}$$

Thus, in d dimensions, $\hat{T}^{\mu\nu} \equiv T^{\mu\nu} - \frac{1}{d}\eta^{\mu\nu}T$ is the traceless version of $T^{\mu\nu}$.

3 Levi-Civita Symbol

12.) The crux of this problem is that the Levi-Civita symbol (also referred to as the Levi-Civita tensor) is fully antisymmetric in its indices:

$$\varepsilon_{\mu\nu\rho\sigma} = -\varepsilon_{\nu\mu\rho\sigma} = -\varepsilon_{\rho\nu\mu\sigma} = -\varepsilon_{\sigma\nu\rho\mu} = -\varepsilon_{\mu\rho\nu\sigma} = -\varepsilon_{\mu\sigma\rho\nu} = -\varepsilon_{\mu\nu\sigma\rho} \, . \tag{3.1}$$

That is, if we swap any two indices in the tensor, we pick up an overall minus sign. This is because the Levi-Civita tensor is defined to be $+1$ for even permutations and -1 for odd permutations. Swapping any two numbers in an even permutation of $\{0, 1, 2, 3\}$ results in an odd permutation, and swapping any two numbers in an odd permutation of $\{0, 1, 2, 3\}$ results in an even permutation. For example, consider the following sequence of swaps:

$$\underbrace{\varepsilon_{0123}}_{\text{even permutation}} \xrightarrow[\text{swap first and third numbers}]{} \underbrace{\varepsilon_{2103}}_{\text{odd permutation}} \, ,$$

$$\underbrace{\varepsilon_{2103}}_{\text{odd permutation}} \xrightarrow[\text{swap second and third numbers}]{} \underbrace{\varepsilon_{2013}}_{\text{even permutation}} \, ,$$

$$\underbrace{\varepsilon_{2013}}_{\text{even permutation}} \xrightarrow[\text{swap first and fourth numbers}]{} \underbrace{\varepsilon_{3012}}_{\text{odd permutation}} \, ,$$

and so on. Thus, by definition, swapping a pair of indices in the Levi-Civita tensor picks up a minus sign, and so the Levi-Civita tensor is antisymmetric in any pair of its indices.

Intuitively, then, if we contract two of its indices with a tensor, only the antisymmetric part of that tensor will be important. We can see this explicitly by first noting that

$$\begin{aligned} \varepsilon_{\mu\nu\rho\sigma}T^{\rho\sigma} &= -\varepsilon_{\mu\nu\sigma\rho}T^{\rho\sigma} \\ &= -\varepsilon_{\mu\nu\rho\sigma}T^{\sigma\rho} \, , \end{aligned} \tag{3.2}$$

where in the first equality we used antisymmetry of $\varepsilon_{\mu\nu\rho\sigma}$ and in the second line we relabeled dummy indices. Then, using the definition from problem 7 of the antisymmetric part of a tensor, we find that

$$\begin{aligned} \varepsilon_{\mu\nu\rho\sigma}T^{[\rho\sigma]} &= \varepsilon_{\mu\nu\rho\sigma}\left(\frac{1}{2}(T^{\rho\sigma} - T^{\sigma\rho})\right) \\ &= \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}T^{\rho\sigma} - \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}T^{\sigma\rho} \\ &= \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}T^{\rho\sigma} + \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}T^{\rho\sigma} \\ &= \varepsilon_{\mu\nu\rho\sigma}T^{\rho\sigma} \, . \end{aligned} \tag{3.3}$$

So, we've proven what we set out to. Note also that it didn't matter which pair of indices in the Levi-Civita tensor were contracted with $T^{\mu\nu}$, since the Levi-Civita tensor is antisymmetric in every single pair of indices. Thus, we can also immediately conclude that

$$\varepsilon_{\mu\nu\rho\sigma} T^{\nu\sigma} = \varepsilon_{\mu\nu\rho\sigma} T^{[\nu\sigma]} \ , \quad \varepsilon_{\mu\nu\rho\sigma} T^{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma} T^{[\mu\nu]} \ , \quad \text{etc.} \tag{3.4}$$

Another quick corrolary we can prove is that $\varepsilon_{\mu\nu\rho\sigma}\eta^{\rho\sigma} = 0$. This is because the metric tensor $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$ are both symmetric, and so their antisymmetric pieces vanish, i.e. $\eta_{[\mu\nu]} = 0$ and $\eta^{[\mu\nu]} = 0$.